Maxwell's Equations is the Most Basic for Satellite Communications

—Its Creation Background and Derivation Procedure—

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1. Introduction

Maxwell's equations is the culmination of electromagnetism and is the very basics of radio communication, of course, is also the basis of satellite communications. However, on another thought, although it may not apply to current students, Maxwell's equations might have been only passed through as a series of operations of a mathematical expression in the lecture of electromagnetics and antenna engineering at the time of my student in the late 1960s. I have noticed that I have not learned properly including the physical significance of Maxwell's equations. As the cause, since Maxwell's equations rests at the end of electromagnetism textbook, it was dispensed easily from time constraints. Maxwell's equations was handled as it had been already learned in the lecture associated with the antenna and radio wave propagation in the next step to the lecture of electromagnetism. Also, for me, there was impression that Maxwell's equations was only a formula including troublesome vector operation such as "rot" and/or "div".

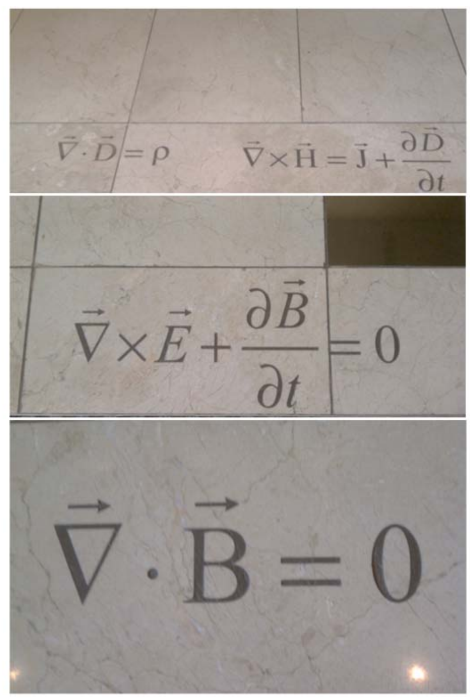

Fig. 1 Maxwell's equations carved into the floor of the National Academies Keck Center lobby, Courtesy of Mr. Maribeth Keitz of the Center.

By the way, recently, there seems to be a discussion of whether mathematics is necessary for the Faculty of Engineering. However, mathematics is the basis of engineering creativity. Especially as a representative of basis, relief of Maxwell's equations is engraved in the marble floor of the U.S. Academy of Engineering [1]. **Figure 1** shows a photograph of a relief carved on the floor that had been sent from a person of the U.S. Academy of Engineering. In addition, a book of "Seventeen Equations That Changed the World" that has recently been published [2][3] treats Maxwell's equations as an important role. The above-mention is a motivation that I decided to relearn Maxwell's equations including its physical significance this time. Therefore, please forgive me that half a century old classic literatures are included as references.

Maxwell derived the Maxwell's equations as a mathematical equation from the result of Faraday's experimental and theoretical study, and he has not only developed the science and technology of the era of that time but also opened a new world. In the following, after describing the historical background that Maxwell's equations was created, the process of deriving the Maxwell's equations is described as a main subject. Then, after Hertz's experiment and Marconi's radio communication experiment are mentioned, calculation of the antenna pattern needed in the radio engineering, and Friis transmission formula are shown, although it might be outside the scope of electromagnetism. Although exact derivation is possible in accordance with the textbook, please forgive me that there is no consistency in how to use the variable in the following induction of the formula and that there may be a place where sacrifices a little rigor, because there were no proper literatures included through all the items.

There seems to be many experts who understand completely all of the Maxwell's equations, thus such persons may think why to relearn it now. Since this report only follows conventional result without any new one, such experts should skip the sections of derivation of a numerical formula. The author will be happy if they are interested in looking back upon rather creation process and background of creating the equations.

2. Time Background of Establishing Maxwell's Equations

Before delving into the subject, we would like to investigate about Faraday who made an opportunity of Maxwell's research. First, it is investigated what is the age of Maxwell's equations' birth. That time is the era of the early 19th century through the 18th century, it is an eye-popping fact that a great scientist is produced one after another.

A magnet is known as remote action for a long time, there has been interest also as a tool of magic as well as the subject of interest. In this regard, there are details in "Yoshitaka Yamamoto: "Discovery of Gravity and Magnetic Force Volume 3" [4], and it is described as follows in the last of the document. "With this, this long story about the old scientific history of magnetic force and gravity is over. Newton's and Coulomb's concept of remote action would be reexamined by a process from Faraday to Einstein before long, but it is a story beyond this book." In other words, Maxwell's equations to be handled here is just "a story beyond this book".

2.1 Century of Great Scientist Producing

In the middle of the 19th century when Maxwell lived, the Industrial Revolution was occurred, and science and technology were developed remarkably. It is surprised that many geniuses who developed science and technology were produced in this era. The times that the geniuses lived in is shown in **Fig. 2** in age order. In Fig. 2, mathematicians including Euler, Laplace, Gauss, Bessel, Stokes, Heaviside,

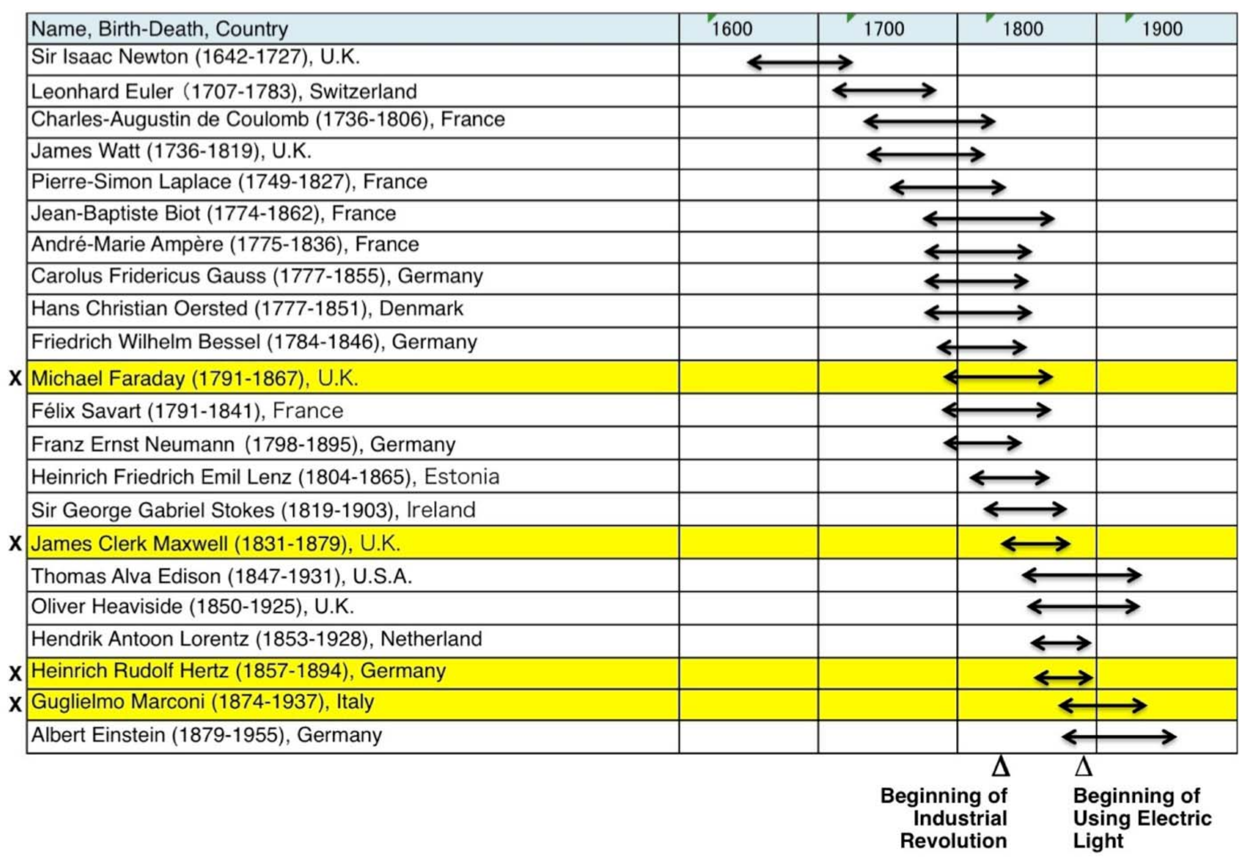

Fig. 2 Age of Maxwell's Equations.

and Lorentz are famous for the formulas that have the name of the genius. Almost all of these formulas are used in a derivation process of Maxwell's equations. The main stream of development of electromagnetism is followed by Coulomb, Biot, Savart, Ampere, Oersted, Faraday, Maxwell, Hertz, and Marconi. And then it led to achievements of Einstein.

Newton, Watt and Edison are also shown for reference in Fig. 2. Newton showed concept of remote action with an equation of gravity, Watt practiced the steam engine that placed at center of the Industrial Revolution that began in the middle of the 18th century, and Edison promoted the use of an electric light, and the use of electric illumination was started in the end of the 19th century.

In the following, attention is paid for achievements of Faraday, Maxwell, Hertz and Marconi as shown by "X" at the left-side end of Fig. 2. First, the background about former two scholars are described.

2.2 Faraday [5]

Michael Faraday (1791-1867), cf. **Fig. 3**, opened up new situation of electromagnetism. He was born as a child of a British smithy, and he became the apprentice to a local bookbinder and bookseller. But he attended an open lecture of the Royal Institution and Royal Society and entreated to work about natural science. He was adopted in 1813 as a laboratory assistant of Royal Society chemist H. Davy. Because the Royal Institution and Royal Society carried out open lectures for the mission of spreading knowledge of useful mechanical invention and of making it introduce to the public at that time, it was able to draw the Faraday's talent into a research. Such an open lecture may be that the present independent administrative research institution in Japan should learn.

Fig. 3 Michael Faraday.

Faraday studied a magnetic field around a direct current, established the basic theory of an electromagnetic field, invented a device (a motor) which uses electromagnetism, and got the basics of the later motor technology. Terms including an anode electrode, cathode electrode and SI unit of "Farad (F)" are associated with Faraday.

Faraday applied an electromagnetism phenomenon the concept of proximity action to lines of electric and magnetic force in space, whereas other scientists regarded as remote action in the dynamics. He discovered that an electric current flowed even if a magnet in a coil of an empty core is moved, that an electric current drifted even if a magnet is fixed again and conducting wire is moved. According to these experiments, he showed that an electric field is produced by a change of a magnetic field. Maxwell derived a mathematical model from this Faraday's electromagnetic induction law later. It becomes one of Maxwell's four equations, and it became the field theory after it was generalized more.

On the other hand, Faraday did not know most of the high mathematics due not to receive higher education, but he is one of scientists who influenced most in history. He took post of the first generation Fuller professor of the Royal Institution.

2.3 Maxwell [6]

James Clerk Maxwell (1831-1879), cf. **Fig. 4**, whose father was a lawyer with a British feudal lord, entered Edinburgh Academy in 1841. In such an academy which was established on a wave of the Industrial Revolution, businessmen, lawyers, engineers, scientists and artists performed free discussion. In 1846, he wrote his first scientific paper about a mechanical means of drawing mathematical curves with a piece of twine, and the properties of ellipses at the age of 15. He was admitted to University of Edinburgh by this paper, but deferred the entrance according to his father's advice that should wait until becoming16 years old.

Fig. 4 James Clerk Maxwell

In 1847, he entered University of Edinburgh. He paid his attention to a study of Faraday's electromagnetism phenomenon during University of Cambridge being on the register roll. He started the study in 1850. In 1856, he replaced an electric and magnetic line of force proposed by Faraday with a streamline of fluid, in his paper on Faraday's line of force. He assumed that electromotive force occurred by electromagnetic induction was a change of time of magnetic flux and expressed it mathematically. A unified theory was demanded in the era when electric technology was developed drastically under expanding capitalism economy including that Atlantic crossing submarine cable was laid.

In 1861, he created a general concept of a displacement current to produce an electric current by displacement of fine particles even if an electric current did not flow like a dielectric, in his paper about a physical lines of force. It seems not to have been understood due to its difficulty, but, in 1864, a basic equation of Maxwell was derived in his paper: "A Dynamical Theory of the Electromagnetic Field". In 1873, he wrote a textbook: "A Treatise on Electricity and Magnetism", and showed that there existed an electromagnetic wave, and the propagation speed was equal to speed of light. This led to the theory of relativity of Einstein later.

It could be said that Maxwell's equations is not appeared alone suddenly, rather it appeared in accordance with the time spirit of a request under the leap of electric technology and the development of science and technology by the Industrial Revolution.

3. Maxwell's Equations [7]

Maxwell's equations become Eq. (1), (2), (3), and (4), if a format same as Fig. 1 is used.

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \quad .... \quad (1)$$

$$\nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} = J \quad .... \quad (2)$$

$$\nabla \bullet \boldsymbol{D} = \rho \quad .... \quad (3)$$

$$\nabla \bullet \boldsymbol{B} = 0 \quad .... \quad (4)$$

where, **E**: Electric field, **H**: Magnetic field, **D**: Electric flux density, **B**: Magnetic flux density, J: Current density, ρ: Charge density. The vector operation is shown in the following, including to use it later. $\nabla\times$: rotation, $\nabla\bullet$: divergence, ∇: gradient. ∇ is called as nabla, and $\nabla\bullet\nabla$ (div(grad)) is described as ∇^2 and is called as Laplacian. These are shown by Eq. (5) and (6).

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \quad .... \quad (5)$$

$$\nabla \bullet \nabla = div(grad) = \nabla^2 = \left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}\right) \quad .... \quad (6)$$

A vector operation is hard to become familiar with it unless to get used, but accepting it, derivation of Maxwell's equations is advanced. The derivation process becomes four steps as follows.

- First step: Static electric field by charge—Gauss's Law in differential form
- Second step: Magnetic field produced around electric current—Biot-Savart Law
- Third step: Electric current produced by magnetic field—Faraday's Law
- Final step: Introduction of displacement current—Completion of Maxwell's equations

3.1 First Step: Static Electric Field by Charge

Force **F** acting on two electric charge q_1 and q_2 each other separated by distance r is measured by using experimental equipment as shown in **Fig. 5**, and it is given in Eq. (7).

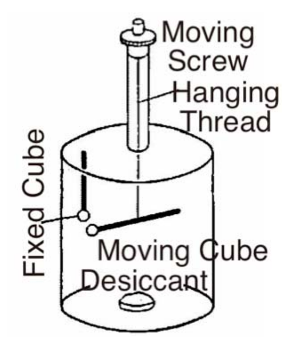

Fig. 5 Coulomb's experiment [8].

$$\boldsymbol{F} = \frac{q_1 q_2}{4\pi\varepsilon_0}\frac{\boldsymbol{r}}{|r|^3} \quad .... \quad (7)$$

where, ε_0 is a dielectric constant of vacuum. As shown below, this force **F** is not produced by charge q_1 and q_2 acted each other (remote action), but the force **F** of Eq. (8) acts to a charge q_2 in the electric field **E** of Eq. (9) produced by a charge q_1 (proximity effect). It is said that Faraday created this idea originally, as mentioned in **2.2**. Its medium was unidentified, but a thought that something should exist developed into the thought that the ether should exist.

$$\boldsymbol{F} = q_2\boldsymbol{E} \quad .... \quad (8)$$

where,

$$\boldsymbol{E} = \frac{q_1}{4\pi\varepsilon_0}\frac{\boldsymbol{r}}{|r|^3} \quad .... \quad (9)$$

For simplicity, consider that a charge q is surrounded by sphere of radius r. According to Gauss's law in integral form, since total in plane surrounding the charge q of the electric field emanating from the charge q equals to q. Eq. (10) is obtained by assuming **n** a unit vector of normal direction.

$$\int \boldsymbol{E} \bullet \boldsymbol{n}\, dS = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \times 4\pi r^2 = \frac{q}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int \rho\, dV \quad ..... \quad (10)$$

According to Gauss's Law (The sphere integral of closed surface is changed to the volume integral of total volume surrounded by closed surface), Eq. (11) is given as follows:

$$\int \boldsymbol{E} \bullet n\, dS = \int \nabla \bullet \boldsymbol{E}\, dV \quad .... \quad (11)$$

Comparing Eq. (10) with Eq. (11), Eq. (12) is obtained. This is called as Gauss's Law in differential form.

$$\nabla \bullet \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \quad .... \quad (12)$$

Because a dielectric constant is defined a ratio of electric flux density **D** to electric field **E**, in the vacuum,

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} \quad .... \quad (13)$$

Taking divergence for both sides of Eq. (13) and comparing it with Eq. (12),

$$\nabla \bullet \boldsymbol{D} = \rho \quad .... \quad (14)$$

Eq. (14) is the third equation of Maxwell's equations.

3.2 Second Step: Magnetic Field Produced Around Electric Current

Clockwise magnetic field B (magnetic flux density) occurs to circumference of a direct current as a flow of a charge for the electric current direction. Therefore, in order to find magnetic flux density by an electric current flowing through the linear conducting wire which has arbitrary form, the electric current is divided into infinitesimal electric current piece, Jds, and then each magnetic flux density by an infinitesimal electric current piece is summed. However, because the electric current is different from charges and has long connection, it is generally difficult to take out only the part and to separate to examine the action. This problem was solved by Biot and Savart and both researchers overcame this difficulty by a good device and found experimental law about magnetic flux density by an infinitesimal electric current piece. The magnetic flux density dB, which is an infinitesimal magnetic field made by electric current J at a remote point of distance r flowing through infinitesimal length ds on conducting wire is given by (Biot-Savart Law, 1820),

$$d\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{J ds \times \boldsymbol{r}}{r^3} \quad .... \quad (15)$$

The next calculation technique is used to pursue **B** from Eq. (15). A vector **A** satisfying Eq. (16) is introduced. This **A** is called as a vector potential.

$$B = \nabla \times A \text{ .... } (16)$$

Although the calculation operation is omitted, the vector potential **A** can be given as Eq. (17) which uses the current density in the case of steady current state.

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\int \frac{J}{|\boldsymbol{r}|}dV \text{ .... } (17)$$

For the arbitrary vector **X**,

$$\nabla \bullet \nabla \times \boldsymbol{X} = 0 \text{ .... } (18)$$

Taking divergence of both sides of Eq. (16) gives Eq. (19).

$$\nabla \bullet \boldsymbol{B} = 0 \text{ .... } (19)$$

Eq. (19) is the fourth equation of Maxwell's equations.

Next, $\nabla \times \boldsymbol{B}$ is calculated.

$$\nabla \times \nabla \times \boldsymbol{X} = \nabla\nabla \bullet \boldsymbol{X} - \nabla^2 \boldsymbol{X} \text{ .... } (20)$$

then, $\nabla \times \boldsymbol{B} = \nabla \times \nabla \times \boldsymbol{A} = \nabla\nabla \bullet \boldsymbol{A} - \nabla^2 \boldsymbol{A}$ .... (21)

The result of calculation of Eq. (21) is given by Eq. (22), if the calculation operation is accepted.

$$\nabla \times \boldsymbol{B} = \mu_0 J \text{ .... } (22)$$

Eq. (22) expresses that an infinitesimal clockwise magnetic field occurs on the circumference when electric current density exists (Ampere's Law in differential form). However, pay attention to that the right side think to be only the stationary electric current which does not change in terms of time, as the condition is shown when the above-mentioned vector potential is expressed (the problem will be described in 3.4). Furthermore, same as an electric field, because the magnetic permeability is defined as a ratio of magnetic flux density **B** to a magnetic field **H**, in the vacuum,

$$\boldsymbol{B} = \mu_0 \boldsymbol{H} \text{ .... } (23)$$

Taking rotation for both sides of Eq. (23) and comparing it with Eq. (22),

$$\nabla \times \boldsymbol{H} = J \text{ .... } (24)$$

Eq. (24) is the second equation of Maxwell's equations.

3.3 Third Step: Electric Current Produced by Magnetic Field

The fact treated in the third step is based on Faraday's idea in 1821 that its reverse should exist, if magnetism is produced by an electric current. In addition, according to Lenz's Law, the induced current in the loop is always in such a direction as to oppose the change that produced. Furthermore, according to Neumann's Law, when the number of magnetic fluxes interlinked with a circuit (the number of lines of magnetic force passing through the coil) is changing, electromotive force is produced in equal to the rate of decrease of the flux linkage.

Suppose the produced electromotive force by ϕ,

$$\phi = -\frac{d\Phi}{dt} \text{ .... } (25)$$

where, Φ is total number of magnetic fluxes interlinked with a circuit, and $-d\Phi/dt$ means a ratio of decreased number of the magnetic flux linkage.

Since the electromotive force ϕ can be obtained by the line integral of the electric field along the path of the coil,

$$\phi = \int \boldsymbol{E} \bullet ds \text{ .... } (26)$$

Magnetic flux passing through the inside of the coil can be expressed by integrating on the surface of the edge of the course of the coil:

$$\Phi = \int \boldsymbol{B} \bullet \boldsymbol{n}\, dS \quad .... \quad (27)$$

Applying Eq. (26) to the left side of Eq. (25), and Eq. (27) to the right side of Eq. (25),

$$\int \boldsymbol{E} \bullet d\boldsymbol{s} = -\frac{d}{dt}\left(\int \boldsymbol{B} \bullet \boldsymbol{n}\, dS\right) \quad .... \quad (28)$$

Taking Stokes's theorem to substitute the surface integral of vector for the line integral (ds to dS) for Eq. (28), Eq. (29) is obtained.

$$\int \nabla \times \boldsymbol{E} \bullet \boldsymbol{n}\, dS = -\int \frac{\partial \boldsymbol{B}}{\partial t} \bullet \boldsymbol{n} dS \quad .... \quad (29)$$

Eq. (30) is obtained from Eq. (29). This is called as Faraday's law of induction.

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \quad .... \quad (30)$$

Eq. (30) is the first equation of Maxwell's equations.

3.4 Final Step: Introduction of Displacement Current

Eq. (30), (24), (14) and (19) obtained from **3.1** to **3.3** are rewritten together as follows:

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \quad .... \quad (31)$$

$$\nabla \times \boldsymbol{H} = J \quad .... \quad (32)$$

$$\nabla \bullet \boldsymbol{D} = \rho \quad .... \quad (33)$$

$$\nabla \bullet \boldsymbol{B} = 0 \quad .... \quad (34)$$

Four equations mentioned above do not completely accord in Maxwell's equations. The reason is because there is a problem in $\nabla \times \boldsymbol{H} = J$ of Eq. (32) obtained in a magnetic field occurring around an electric current. The reason is as follows. If divergence is applied for both sides of Eq. (32), Eq. (35) is obtained.

$$\nabla \bullet \nabla \times \boldsymbol{H} = \nabla \bullet \boldsymbol{J} = 0 \quad .... \quad (35)$$

This is correct in a steady state, but is not generally correct when charge density in space changes with time. A charge stored up by a condenser drifts to low electrical potential if an electric wire made an escape way as a cause of letting occur a change, for example. In this case, an electric current occurs from a source that charges are accumulated. Then the accumulated charges decrease in accordance with an electric current flow. Thus, from a condition of charge immortality in an arbitrary point, Eq. (36) should be established.

$$\nabla \bullet \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0 \quad .... \quad (36)$$

The following handling is performed to reflect this [8]. At first, differentiating both sides of Eq. (33),

$$\nabla \bullet \frac{\partial \boldsymbol{D}}{\partial t} = \frac{\partial \rho}{\partial t} \quad .... \quad (37)$$

Using Eq. (36),

$$\nabla \bullet \frac{\partial \boldsymbol{D}}{\partial t} + \nabla \bullet \boldsymbol{J} = \nabla \bullet \left(\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}\right) = 0 \quad .... \quad (38)$$

It is clear that $\partial \boldsymbol{D}/\partial t$ exists in the same form as electric current density J in Eq. (38). Thus thinking that the quantity of $\partial \boldsymbol{D}/\partial t$ has a work to make a magnetic field in the same way as a normal electric current and substitute it to the right side of Eq. (32),

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \quad .... \quad (39)$$

The $\partial \boldsymbol{D}/\partial t$ means that a magnetic field could be generated only by an electric field without a (steady) electric current, although it has studied that an electric current created a magnetic field and a change of a magnetic field produced an electric field. This quantity $\partial \boldsymbol{D}/\partial t$ is called as "a displacement current", and it was an idea of Maxwell. By this idea, Maxwell has completed his formula, and classic electromagnetism including propagation of an electromagnetic wave system and other rich contents was completed here.

To regard $\partial \boldsymbol{D}/\partial t$ as one of electric currents was demanded inevitably for completing the idea of field by applying a relation of $\nabla \times \boldsymbol{H} = J$ to the magnetic field without giving a magnetic field from Biot-Savart law. However, this is indeed one of hypothesis, and it is not demonstrated without a reliable experiment. In order to that, it is not except that it is experimentally demonstrated by finding an effect appeared for the first time by having put this clause in it. The typical one of such phenomena is an electromagnetic wave. A change of an electric field at a certain place produces a displacement current and produces a magnetic field by it, and a change of the magnetic flux produces an electric field by induced electromotive force again, and it is an electromagnetic wave that the action reaches in sequence in this way. This is convenient to transmit an electromagnetic field relatively far away. Such an electromagnetic wave was demonstrated by Hertz. This is a great victory of an electromagnetic field theory to the remote action theory [8].

4. Hertz's Experiment [9]

German physicist Heinrich Rudolf Hertz (1857-1894) (cf. **Fig. 6**) clarified Maxwell's electromagnetism theory more and developed it. He made the equipment which generated the electromagnetic wave as shown in **Fig. 7**, and detected existence of emission of an electromagnetic wave in 1888 and demonstrated it by an experiment of distance of 12 m.

Fig. 6 Heinrich Rudolf Hertz

It is said that Hertz did not understand practical value of his experiment. "It is no useful ..... the experiment merely proved that Maxwell was right". When he was asked about the future of the discovery, he answered "Probably there is nothing".

According to the description above, it might be merely thought that Hertz conducted his experiment. However, he observed that strength of an electric field was weakened in inverse proportion to distance, and contributed to establishment of photoelectric effect by paying attention to lost electric charge when ultraviolet rays hit it. In addition, he confirmed that an electromagnetic wave is a transverse wave, propagates at limited speed (velocity of light), and has properties such as propagating through the various materials, a reflection, refraction, deflection the same as light. Furthermore, he performed contribution to fix Maxwell's equations in refreshing form as describe in **7**.

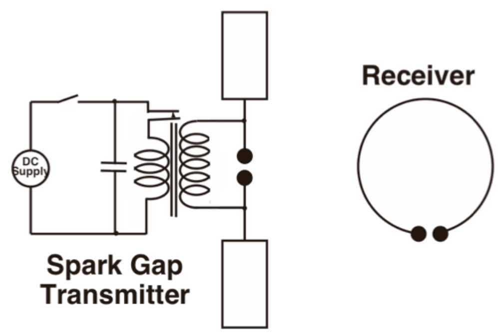

Fig. 7 Hertz's experimental equipment

5. Marconi's Practical Realization of Radio Communication [10][11]

Guglielmo Marconi (1874-1937) (cf. **Fig. 8**) succeeded in the first radio communication in the world in 1895. His experimental device is exhibited at Marconi museum (his birthplace) located at about 15km south from Bologna as shown in **Fig. 9**.

Fig. 8 Guglielmo Marconi.

The experimental device is shown in **Fig. 10**. The transmitter (spark gap transmitter) and receiver (coherer wave detector) themselves are not originally created by Marconi, but an antenna in top and the ground earth bottom are originally developed by him. This antenna and earth transfers an electric wave to the air to jump out of a laboratory, and to reach over an obstacle in the way which seems to be a hill in front of the museum (cf. **Fig. 11**), and long-distance communication of 1.5 km was succeeded.

His result was not evaluated in those days in Italy and moved to the U.K. that was his mother's country because rather the U.K. evaluated his result and he could make the basics of development afterward. It is clarified that the reason why his experimental result has been accepted by the world was based on his parents' eager support as well as on his family's wealth [11]. Marconi won Nobel Prize in Physics in 1909.

Fig. 9 Marconi Museum.

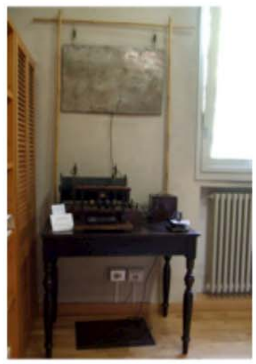

Fig. 10 Radio Communication Experiment Equipment.

Fig. 11 Hillside yard in front of the museum where Marconi conducted his world first experiment.

6. Derivation of Wave Equation from Maxwell's Equations [7][12]

In the radio engineering, quantities varying sinusoidally with time can be represented by complex quantities in the sense that only the real (or imaginary) part has physical significance. An electric field is given by

$$\boldsymbol{E} = |\boldsymbol{E}|\cos(\omega t + \phi) \quad \text{...............} \quad (40)$$

From Euler's formula,

$$e^{\pm jx} = \cos x \pm j\sin x \quad \text{................} \quad (41)$$

Taking real part,

$$\boldsymbol{E} = |\boldsymbol{E}|e^{j\omega t} \quad \text{.................} \quad (42)$$

Then $\partial/\partial t$ can be converted by $j\omega$ [7].

From the first equation of Maxwell's equations (Eq. (1)), the second one (Eq. (2)), $\boldsymbol{B} = \mu\boldsymbol{H}$ and $\boldsymbol{D} = \varepsilon\boldsymbol{E}$,

$$\nabla \times \boldsymbol{E} + j\omega\mu\boldsymbol{H} = 0 \quad \text{.................} \quad (43)$$

$$\nabla \times \boldsymbol{H} - j\omega\varepsilon\boldsymbol{E} = J \quad \text{.................} \quad (44)$$

Taking rotation to both sides of Eq. (43), and discriminating **H** from Eq. (43) by using Eq. (44),

$$\nabla \times \nabla \times \boldsymbol{E} - k^2\boldsymbol{E} = -j\omega\mu J \quad \text{..................} \quad (45)$$

where, $k^2 = \omega^2\varepsilon\mu$ .... (46)

Taking rotation to both sides of Eq. (44), and discriminating **E** from Eq. (44) by using Eq. (43),

$$\nabla \times \nabla \times \boldsymbol{H} - k^2\boldsymbol{H} = \nabla \times J \quad \text{.................} \quad (47)$$

Using Eq. (20) and considering the third equation of Maxwell's equations (Eq. (3)), the fourth one (Eq. (4)),

$$\begin{cases} \nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = j\omega\mu J + \dfrac{1}{\varepsilon}\nabla\rho \quad \text{.........} \quad (48) \\ \nabla^2 \boldsymbol{H} + k^2 \boldsymbol{H} = -\nabla \times J \quad \text{.................} \quad (49) \end{cases}$$

where, at the point without electric current source and electric charge,

$$\begin{cases} \nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = 0 \quad \text{.................} \quad (50) \\ \nabla^2 \boldsymbol{H} + k^2 \boldsymbol{H} = 0 \quad \text{.................} \quad (51) \end{cases}$$

are obtained. These are called as the wave equation.

7. Electromagnetic Field of Plane Wave [13][14]

A plane wave is the wave that an aspect whose phase of a wave accords is plane. Generally speaking, giving electric current source **J**, the electromagnetic field **E** and **H** are obtained through a course using vector potential **A** or Hertz potential **Π** rather than it does direct integral calculus as shown in **Fig. 12** [13].

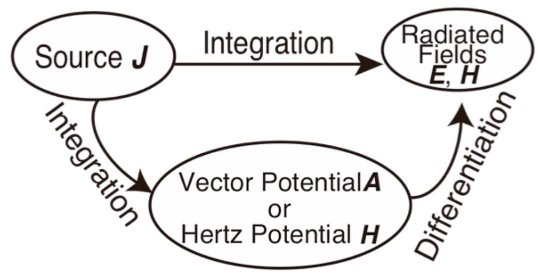

Fig.12 Computing radiated fields from electric sources [13].

Considering a vector function **A** satisfied the following partial differential equation,

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu J \quad \text{.................} \quad (52)$$

Furthermore,

$$\prod = \frac{A}{j\omega\varepsilon\mu} \quad \text{....................} \quad (53)$$

is called as a Hertz potential. Pursue **Π** satisfied the following equation.

$$\nabla \bullet \nabla \prod + k^2 \prod = jJ/\omega\varepsilon \quad \text{.............} \quad (54)$$

Since a plane wave does not have any electric current in the finite region, it can be assumed as ***J****=0*. The solution of electromagnetic field is

$$\begin{cases} E_x = E_0 e^{-jkz} \quad \text{.............................} \quad (55) \\ H_y = H_0 e^{-jkz} \quad \text{...........................} \quad (56) \\ E_y = E_z = H_x = H_z = 0 \quad \text{...........} \quad (57) \end{cases}$$

The electromagnetic wave is a wave accompanied with an electric field and a magnetic field. This is the important conclusion of Maxwell's equations assumed displacement current introduction. The instantaneous value E_{xt} and H_{yt} of E_x and H_y respectively are

$$\begin{cases} E_{xt} = \sqrt{2}\left|E_0\right|\cos(\omega t + \theta - kz) \quad \text{...............} \quad (58) \\ H_{yt} = \sqrt{2}\left|H_0\right|\cos(\omega t + \theta - kz) \quad \text{.............} \quad (59) \end{cases}$$

The result is shown in **Fig. 13**, where λ means wavelength.

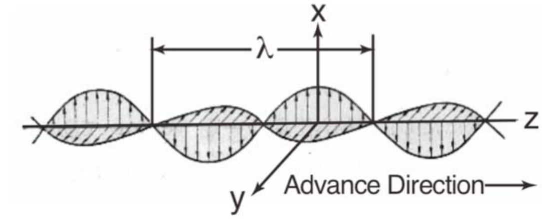

Fig. 13 Instantaneous value of plane wave.

8. From Maxwell's Equations to Special Theory of Relativity [7]

The propagation speed of a plane wave is obtained as follows. In expression of instantaneous value E_{xt} and H_{yt}, of E_x and H_y, an electric field and a magnetic field are constant at the point where $\omega t+\theta-kz$ is constant. Such a point moves to a z direction with time and the speed v is given by

$$v=\frac{dz}{dt}=\frac{\omega}{k}=\frac{1}{\sqrt{\mu\varepsilon}} \quad \text{..............} \quad (60)$$

This speed is equal to velocity of light in vacuum by Eq. (61). The discovery that speed of an electromagnetic wave is equal to velocity of light is developed to the special theory of relativity.

$$v=\frac{1}{\sqrt{\varepsilon_0\mu_0}}=\frac{1}{\sqrt{\left(8.855\times10^{-12}\left[F/m\equiv As/V/m\right]\right)\times\left(4\pi\times10^{-7}\left[H/m\equiv Vs/A/m\right]\right)}}$$
$$=2.996\times10^{8}\left[m/s\right] \quad \text{..} \quad (61)$$

where As/V/m means Ampere•Second/Volt/Meter, and Vs/A/m means Volt•Second/Ampere/Meter.

An original textbook of Maxwell himself "A Treatise on Electricity and Magnetism" in 1873 included the expression of electromagnetic potential [15]. In 1884, Heaviside applied a method of vector analysis and rewrote it to the current form that is easy to look at [16]. Furthermore, in 1890, Hertz discussed the Maxwell's theory configuration again and showed the formula that is erased electromagnetic potential. Since equation system was organized by these activities, it was led that an electric field and a magnetic field were unified (an electromagnetic field) and that light is electromagnetic waves. Most of physicists through the 19th century latter half thought that the equations were merely an approximation to an electromagnetic field, because it was a strange prediction that speed of light is unchangeable for all observers in Maxwell's equations and that it contradicted an exercise law of the Newton dynamics. However, because Einstein submitted the special theory of relativity in 1905, it was clarified that Maxwell's equations are correct, and that the Newton dynamics should be revised. Einstein declared later the origin of the special theory of relativity is an electromagnetic field equation of Maxwell.

9. Electromagnetic Field at Far Field [13]

When an electric current source J_0 and a magnetic current source J_{0m} exist within a medium of no loss, electromagnetic field guided by them is obtained. In this case; Hertz potential $\prod$ and $\prod_m$ are

$$\begin{cases}\nabla^2\Pi+k^2\Pi=\dfrac{jJ_0}{\omega\varepsilon} \quad \text{..................} \quad (62)\\[2ex] \nabla^2\Pi_m+k^2\Pi_m=\dfrac{jJ_{0m}}{\omega\mu} \quad \text{.............} \quad (63)\end{cases}$$

The solution for these equations are given by the following.

$$\begin{cases}\Pi(x,y,z)=\dfrac{1}{j4\pi\omega\varepsilon}\displaystyle\int_V J_0(x',y',z')\frac{e^{-jkr}}{r}dv \quad \text{.............} \quad (64)\\[2ex] \Pi_m(x,y,z)=\dfrac{1}{j4\pi\omega\varepsilon}\displaystyle\int_V J_{0m}(x',y',z')\frac{e^{-jkr}}{r}dv \quad \text{........} \quad (65)\end{cases}$$

The electric field and magnetic one are obtained by calculating these equations,

$$\begin{cases}\mathrm{E}(x,y,z)=\dfrac{1}{4\pi}\displaystyle\int_V\left\{-j\omega\mu\,\mathrm{J}_0\frac{e^{-jkr}}{r}-J_{0m}\times\nabla'\left(\frac{e^{-jkr}}{r}\right)+\frac{\rho}{\varepsilon}\nabla'\left(\frac{e^{-jkr}}{r}\right)\right\}dv \quad \text{.......} \quad (66)\\[2ex] \mathrm{H}(x,y,z)=\dfrac{1}{4\pi}\displaystyle\int_V\left\{-j\omega\varepsilon\,\mathrm{J}_{0m}\frac{e^{-jkr}}{r}+J_0\times\nabla'\left(\frac{e^{-jkr}}{r}\right)+\frac{\rho_m}{\mu}\nabla'\left(\frac{e^{-jkr}}{r}\right)\right\}dv \quad \text{.....} \quad (67)\end{cases}$$

where, ∇' represents differentiation of (x',y',z'), and r is given by

$$r=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \quad \text{..........} \quad (68)$$

When electric current source and magnetic current source are limited to only part of space, an electromagnetic field of only far-zone field is given as follows,

$$\begin{cases} \mathrm{E}\approx \dfrac{e^{-jkR}}{R}\mathrm{D} \quad \text{.....................} \quad (69) \\ \mathrm{H}\approx \dfrac{1}{Z_0}\dfrac{e^{-jkR}}{R}\mathrm{R}_0\times\mathrm{D} \quad \text{.........} \quad (70) \end{cases}$$

where, pay attention that "D" does not mean electric flux density, but is defined as follow:

$$\begin{cases} \mathrm{D}=\left(\mathrm{R}_0\times\mathrm{D}_1+Z_0\mathrm{D}_{1m}\right)\times\mathrm{R}_0 \quad \text{......} \quad (71) \\ Z_0=\sqrt{\mu/\varepsilon} \quad \text{................................} \quad (72) \end{cases}$$

where,

$$\begin{cases} \mathrm{D}_1=-\dfrac{j\omega\mu}{4\pi}\displaystyle\int_V \mathrm{J}_o e^{jk\xi\cos\gamma}dv \quad \text{..............} \quad (73) \\ \mathrm{D}_{1m}=-\dfrac{j\omega\mu}{4\pi}\displaystyle\int_V \mathrm{J}_{om} e^{jk\xi\cos\gamma}dv \quad \text{..........} \quad (74) \end{cases}$$

Variables ξ and γ are introduced to expand e^{-jkr}/r to a form including R.

Electromagnetic field emitted in far field from an arbitrary antenna put in the vicinity of the origin of polar coordinates is expressed by Eq. (69) and (70). When only an electric field is rewritten,

$$\mathrm{E}(R,\theta,\phi)\approx\frac{e^{-jkR}}{R}\mathrm{D}(\theta,\phi) \quad \text{.............} \quad (75)$$

where, $D(\theta,\phi)$ is called as a directivity coefficient and its drawing is called as a radiation pattern. Eq. (75) expresses that an electric field emitted in far zone decreases only in inverse proportion to distance R. This guarantees that the radio wave propagates to very far distance.

9.1 Example of Radiation Pattern: Infinitesimal Dipole [13]

The electromagnetic field from an infinitesimal dipole is calculated, whose length is l as shown in **Fig. 14**. The relationship between charge q and electric current i is given by

$$i=\frac{\partial q}{\partial t} \quad \text{.................} \quad (76)$$

Therefore, when charge varies with angular frequency ω, charge and current are obtained in the same way as Eq. (42) by assuming that maximum value of electric current and charge be I and Q respectively,

$$\begin{cases} q=Qe^{j\omega t} \quad \text{................} \quad (77) \\ i=Ie^{j\omega t} \quad \text{..................} \quad (78) \end{cases}$$

Applying equations above to Eq. (76), Eq. (79) is obtained.

$$I=j\omega Q \quad \text{..................} \quad (79)$$

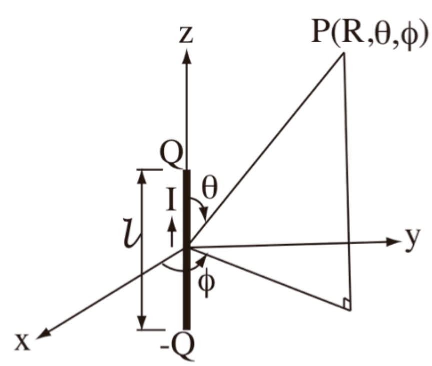

Fig. 14 Infinitesimal dipole.

In this case, a Hertz potential is

$$\Pi_z(x,y,z)=\frac{p}{4\pi\varepsilon}\frac{e^{jkR}}{R} \quad \text{............} \quad (80)$$

where, $\begin{cases} p = Ql \text{ ....} & (81) \\ R = \sqrt{x^2 + y^2 + z^2} \text{ ....} & (82) \end{cases}$

The solution of this equation includes 1/R, 1/R^2 and 1/R^3. The far-zone radiation field is represented by using l instead of p and by assuming $kR >> 1$,

$$\begin{cases} E_\theta = jkIl \dfrac{1}{4\pi}\sqrt{\dfrac{\mu}{\varepsilon}}\dfrac{e^{-jkR}}{R}\sin\theta \text{ ....} & (83) \\ H_\phi = \dfrac{jkIl}{4\pi}\dfrac{e^{-jkR}}{R}\sin\theta \text{ ....} & (84) \end{cases}$$

In case of vacuum, using ε_0, μ_0 of Eq. (61),

$$\frac{1}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 30 \text{ .... } (85)$$

Further, $c = 1/\sqrt{\varepsilon_0 \mu_0}$ is velocity of light in Eq. (61). Thus, assuming λ to be wavelength, k in Eq. (46) is given by

$$k = \frac{2\pi}{\lambda} \text{ .... } (86)$$

Then Eq. (83) is given by

$$E_\theta \approx j60\pi Il \frac{e^{-jkR}}{\lambda R}\sin\theta \quad [V/m] \quad (kR \rangle\rangle 1) \text{ ..... } (87)$$

An example of the radiation pattern of electric field is shown in **Fig. 15** by using Mathematica software mentioned below.

Mathematica program

```
f = RevolutionPlot3D[{Sin[t]}, {t, 0, 2π}, {q, -0.75 π, 0.75 π}, Mesh → 50];
Show[f, Boxed  False, Axes → None, Lighting → "Neutral"]
```

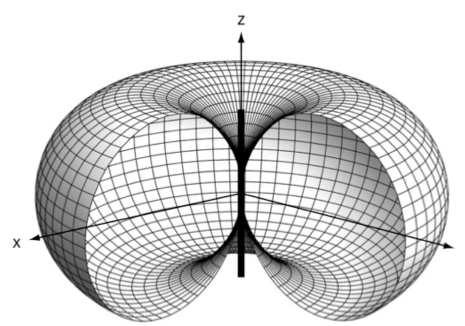

Fig. 15 Radiation pattern of infinitesimal dipole.

9.2 Example of Radiation Pattern: Parabolic Antenna [14]

The radiation pattern of uniform distributed linear array, as shown in **Fig. 16**, with space d is given as follows in the case of broadside array.

$$D(\theta) = 1 + e^{jkd\cos\theta} + e^{jk2d\cos\theta} + \cdots + e^{jk(n-1)d\cos\theta}$$

$$= \sum_{s}^{n-1} e^{jskd\cos\theta} = \frac{\sin\left(\dfrac{n\pi d}{\lambda}\cos\theta\right)}{n\sin\left(\dfrac{\pi d}{\lambda}\cos\theta\right)} \text{ .... } (88)$$

In the case of a parabolic antenna with its aperture diameter of D, a directivity coefficient $D(\theta)$ is given by Eq. (89), considering that uniform illuminated array is set on infinite sheet as shown in **Fig. 17** and applying Huygens' principle (for more detail, refer Ref. [14]).

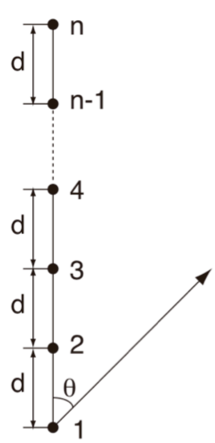

Fig. 16 Linear array.

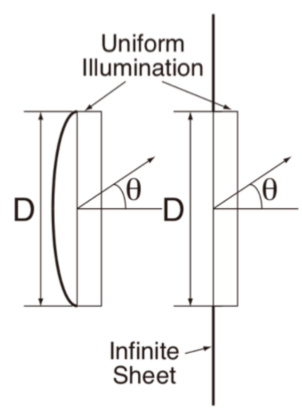

Fig. 17 Parabolic antenna.

$$D(\theta)=\frac{2\lambda}{\pi D}\frac{J_1\left[(\pi D/\lambda)\sin\theta\right]}{\sin\theta} \quad ...... \quad (89)$$

where, J_1 is the first-order Bessel function.

An example of the radiation pattern of parabolic antenna is shown in **Fig. 18** by using Mathematica software mentioned below.

Mathematica program

```
h1 = 7; h2 = Sin[θ]; h3 = 0.11;
g = (BesselJ[1, Sin[h1*h2]]/(h1*h2));
f = ParametricPlot3D[{g*Sin[θ] Cos[ϕ], g*Sin[θ] Sin[ϕ], g},
{θ, -π/2,      π/2}, {ϕ, 0, π}, Boxed -> False, Axes -> False,
Ticks -> None,    Lighting -> "Neutral",   Mesh -> 10];
Show[f, PlotRange -> {{h3, -h3}, {h3, -h3}, {-0.1, 1}} ]
```

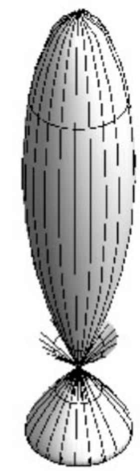

Fig.18 radiation pattern of parabolic antenna.

10. Friis Transmission Formula [7][13][17]

Finally, Friis transmission formula is derived. The power density in the far-field is given by using Eq. (72) and (83),

$$P(R,\theta,\phi)=\frac{\left|E(R,\theta,\phi)\right|^2}{Z_0} \quad .... \quad (90)$$

The radiated power from an antenna W_t is obtained by integrating power in the infinitesimal area of $(R\sin\theta d\phi)(Rd\theta)$ on the spherical surface as shown in **Fig. 19**

$$W_t=\int_0^{2\pi}\int_0^{\pi}P(R,\theta,\phi)(R\sin\theta d\phi)(Rd\theta) \quad .... \quad (91)$$

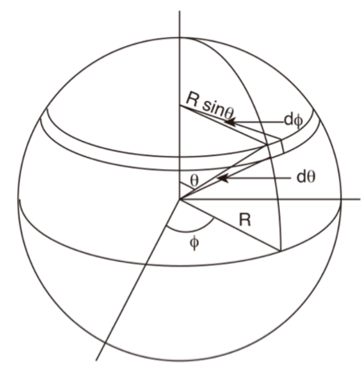

Fig. 19 Sphere including antenna.

Substituting Eq. (86) and (87) to Eq. (91) is given by

$$W_t=R^2\int_0^{2\pi}d\phi\int_0^{\pi}\frac{k^2I^2l^2\frac{1}{(4\pi)^2}\frac{\mu_0}{\varepsilon_0}\frac{\left|e^{-jkR}\right|^2}{R^2}\sin^2\theta}{\sqrt{\frac{\mu_0}{\varepsilon_0}}}\sin\theta d\theta$$

$$=\left(\frac{2\pi}{\lambda}\right)^2I^2l^2\frac{1}{(4\pi)^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\int_0^{2\pi}d\phi\int_0^{\pi}\sin^3\theta\, d\theta$$

$$=\sqrt{\frac{\mu_0}{\varepsilon_0}}|I|^2\left(\frac{l}{\lambda}\right)^2\int_0^{2\pi}\left[-\frac{1}{3}\cos\theta\left(\sin^2\theta+2\right)\right]_0^{\pi}d\phi$$

$$=\frac{2\pi}{3}\sqrt{\frac{\mu_0}{\varepsilon_0}}|I|^2\left(\frac{l}{\lambda}\right)^2 \quad .... \quad (92)$$

Looking at this radiation phenomenon from the transmitting point, the power of W_t is consumed by the feed current of I. Therefore, since a resistance R_t is loaded effectively, Eq. (93) is obtained.

$$R_t=\frac{W_t}{|I|^2}=\frac{2\pi}{3}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left(\frac{l}{\lambda}\right)^2 \quad .... \quad (93)$$

This R_t is called as a radiation resistance.

On the other hand, when a voltage V_r is induced at a terminal of the receiving antenna in the receiving side, the current I flowing through the load is given by an arbitrary load impedance Z_i and receiving antenna impedance Z,

$$I = \frac{V_r}{Z + Z_l} \quad \text{.................} \quad (94)$$

Since the maximum power that can be taken is obtained when Z_i is a complex conjugate of Z, assuming the real part of Z_i and Z to be R_r, the current I is

$$I = \frac{V_r}{2R_r} \quad \text{..................} \quad (95)$$

The maximum power that can be taken W_t is

$$W_r = I^2 R_r = R_r \left| \frac{V_r}{2R_r} \right|^2 = \frac{|V_r|^2}{4R_r} \quad \text{.............} \quad (96)$$

Considering incident radio wave from direction θ to the enough short dipole whose length is l comparing wavelength as shown in **Fig. 20**, the receiving voltage is

$$V_r = El \sin\theta \quad \text{...................} \quad (97)$$

Because it is possible to put the input resistance R_r is equal to the radiation resistance R_t if the internal loss of antenna is zero, Eq. (98) is obtained by substituting Eq. (93) and (97) to Eq. (96).

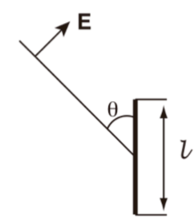

Fig. 20 Infinitesimal dipole for reception.

$$W_r = \frac{3\lambda^2}{8\pi} \sqrt{\frac{\varepsilon_0}{\mu_0}} |E|^2 \sin^2\theta \quad \text{........} \quad (98)$$

Because the directivity gain of infinitesimal dipole, G_r, is calculated to be $G_r = (3/2)\sin^2\theta$, Eq. (98) is

$$W_r = \frac{\lambda^2}{4\pi} \sqrt{\frac{\varepsilon_0}{\mu_0}} G_r |E|^2 \quad \text{............} \quad (99)$$

Changing a viewpoint here, if a signal with the power density P per unit surface, given by Eq. (90), is received, and received power W_r is obtained by transmitting to the receiving cross section A_r, Eq. (100) is obtained by using Eq. (90) and (99).

$$A_r = \frac{W_r}{P} = \frac{\lambda^2 G_r}{4\pi} \quad \text{...............} \quad (100)$$

A_r is called as an effective aperture of antenna.

If transmited output is W_t and transmit antenna gain is G_t, power density P_r at distance R is

$$P_r = \frac{W_t G_t}{4\pi R^2} \quad \text{...............} \quad (101)$$

Assuming receive antenna gain to be G_r, received power W_r is obtained as Eq. (102) by changing P of Eq. (100) to P_r of Eq. (101).

$$W_r = \frac{\lambda^2 W_t G_t G_r}{(2\pi R)^2} \quad \text{...........} \quad (102)$$

This is called as Friis transmission formula that Harald T. Friis of Bell Lab. derived in 1945[17].

11. Conclusions

The introduction of Maxwell's equations and related wireless formulas will be ended. As can be seen from the induction of equations, sophisticated mathematical technique has been used for the derivation process, although it appears to be simple looking at the only result. Almost all the results by mathematician and scientist that were produced in the 19th century as shown in Fig. 2 have been used. On the contrary, it is necessary to note that Maxwell's equations appeared just by such many genius talents. If our country aims at innovation of technology, it is just important to ascertain what the spirit of the age is and what becomes motivation, and to produce the system helping it in future.

Even a recent newspaper said that Maxwell's equations is an example of importance of basic science to suffer about 100 years till it becomes useful for society [18]. Also, it is described about utility of a Friday lecture of the Royal Research Institute which still continues [19].

The author had felt that Maxwell's equations only passed me by as a part of engineering education until now, but when I have relearned again the background-derivation process as mentioned above, I have felt that I was given a great foundation of communication including satellite communications. Is this only me?

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